*J. Fluid Mech.* (1986), *vol.* 168, *pp.* 501–513
*Printed in Great Britain*



# Gravity and shear wave stability of free surface flows. Part 1. Numerical calculations

By R. W. CHIN†, F. H. ABERNATHY AND J. R. BERTSCHY‡
Division of Applied Sciences, Harvard University, Cambridge, Massachusetts 02138

(Received 25 April 1984 and in revised form 9 January 1986)

The linear, two-dimensional stability of flows down an inclined plane has been examined at large Reynolds numbers. Both the surface and shear wave modes have been numerically investigated, involving changes in angle of inclination, surface tension, and form factor.

## 1. Introduction

The stability of the free surface flow down an inclined plane has been of interest for a number of years. The main concern has been low Reynolds number (*Re* based on surface velocity $U_s$, and depth $H$) flows which are important in coating processes. Yih (1963) correctly formulated the problem for small disturbances, and obtained the critical Reynolds number $Re_c$ as $\frac{5}{4}\cot(\theta)$ (where $\theta$ is the angle of inclination) for the surface mode of the parabolic, mean-velocity profile at zero surface tension. This gravity wave travelled at a speed greater than the surface velocity, and at low *Re*, controlled the overall stability of the flow. Graef (1966) used expansions in various functions to analytically determine constant-growth curves for different values of $\theta$ and surface tension $\sigma$ for $Re < 150$. Pierson & Whitaker (1974) numerically calculated stabilizing effects of surface tension for the vertical film flow for $Re < 1000$. As *Re* and $\sigma$ increased, Anshus (1972) extracted a number of asymptotic results.

For large-*Re* flows, a growing shear mode is also possible, travelling at fluid-particle speeds. Lin (1967) analytically solved for the neutral growth curve. However, an incorrect surface boundary condition was used. De Bruin (1974) numerically solved the correctly formulated problem, involving zero surface tension, but varying angle of inclination.

Present interest in the inclined plane geometry arises from polymer-flow experiments on a wide water channel (large flow width to depth ratio) as described in Bertschy, Chin & Abernathy (1983), Abernathy *et al.* (1980), and Bertschy & Abernathy (1977). However, some missing information on the stability of the inclined flow was needed as a basis of comparison to the polymer-flow experiments.

The objective of this paper, therefore, is to extend previous numerical results on the stability of the inclined plane flow. Surface-tension effects will be investigated at large *Re*. Form factor FF (displacement thickness/momentum thickness) effects will also be calculated because of the interest in developing flows on a water channel. Both gravity and shear modes will be considered. Section 2 summarizes the analysis, while §3 discusses the numerical results. Measurements of shear-wave characteristics will be compared with theory in a future paper.

† Present address: Shell Development Company, Westhollow Research Center, P.O. Box 1380, Houston, Texas 77001.
‡ Present address: IBM, G.P.D. Lab., Dept. F-44005, 5600 Cottle Road, San Jose, Calif. 95193.

## 2. Stability analysis

The present calculations will consider the linear stability of the streamwise ($x$), mean-velocity profile $\overline{U}(y)$, where $y$ is the vertical coordinate. The wall is located at $y = 0$, and the free surface at $y = 1$. All velocities, distances and time have been non-dimensionalized by $U_s$, $H$ and $H/U_s$ respectively.

The fluctuation velocities $u$ and $v$ in the $x$- and $y$-directions respectively, are obtained from the stream function $\psi(x, y)$, as

$$u(x,y) = \frac{\partial\psi}{\partial y}, \quad v(x,y) = -\frac{\partial\psi}{\partial x},$$

with

$$\psi(x,y,t) = \phi(y)\,e^{i\alpha(x-ct)},$$

where $\alpha$ is the wave number, $c$ the wave velocity, and $t$ the time. Two-dimensional disturbances are considered because Squire's (1933) theorem is valid for the present free-surface flow.

The appropriate stability equation for infinitesimal disturbances is the well-known Orr–Sommerfeld eigenvalue equation,

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha\,Re([(\overline{U}-c)(\phi''-\alpha^2\phi) - \overline{U}''\phi], \tag{1}$$

where $\phi(y)$ is the amplitude of the stream function, and the $'$ denotes differentiation with respect to $y$, i.e. $\partial/\partial y$. The boundary conditions are

$$\phi(0) = 0, \tag{2a}$$

$$\phi'(0) = 0, \tag{2b}$$

$$i\phi''' + \phi'[-3i\alpha^2 + \alpha\,Re(\overline{U}-c) - \frac{\alpha\phi[2\cot\phi + \alpha^2\gamma(Re^2\sin\phi)^{-\frac{1}{3}}]}{\overline{U}-c} = 0 \quad \text{at } y = 1, \tag{3}$$

and

$$\phi'' + \phi\left[\alpha^2 - \frac{\overline{U}''}{\overline{U}-c}\right] = 0 \quad \text{at } y = 1. \tag{4}$$

Equations ($2a$, $b$) are the no-slip conditions at the wall, $y = 0$. Equations (3) and (4) represent the normal and zero-shear conditions at the free surface, $y = 1$. The parameter $\gamma \equiv (\sigma/\rho)[2/(g\nu^4)]^{\frac{1}{3}}$ (where $\rho$ is the density, $\nu$ the kinematic viscosity, and $g$ the gravitational constant) controls the effect of surface tension, and is approximately 4300 for water at room temperature. Temporal stability will be considered, i.e. $\alpha \equiv \alpha_T$ is real, and $c \equiv c_T = c_{Tr} + ic_{Ti}$ is complex. Growth (decay) is then determined by $c_{Ti} > 0$ ($c_{Ti} < 0$). The neutral curve is determined by $c_{Ti} = 0$. For comparison with experiment, one should consider spatial decay where the wavenumber $\alpha \equiv \alpha_S = \alpha_{Sr} + i\alpha_{Si}$ is complex, and the frequency, $\alpha c \equiv \omega_S$, is real. Gaster's (1962) transformation is then used to relate spatial and temporal parameters, i.e. $\omega_S = \alpha_T c_{Tr} \equiv \omega$, $\alpha_{Sr} = \alpha_T$, and $\alpha_{Si} = -\alpha_T c_{Ti}/c_g$, where $c_g$ is the group velocity. $c_g$ is determined from the slope of the frequency–wavenumber relationship as $\partial\omega/\partial\alpha_T$.

The family of mean-velocity profiles to be considered is

$$\overline{U}(y) = 2y - y^2 + a(-4y + 11y^2 - 9y^3 + y^4 + y^5). \tag{5}$$

The above satisfies $\overline{U}(0) = 0$, $\overline{U}(1) = 1$, and $\overline{U}''(1) = -2$. The last requirement is made in order that the free-surface boundary conditions remain unchanged. When $a = 0$, one recovers the asymptotic, parabolic velocity profile. The above was chosen to simulate the developing flow on a water channel (Bertschy *et al.* 1983). Although the

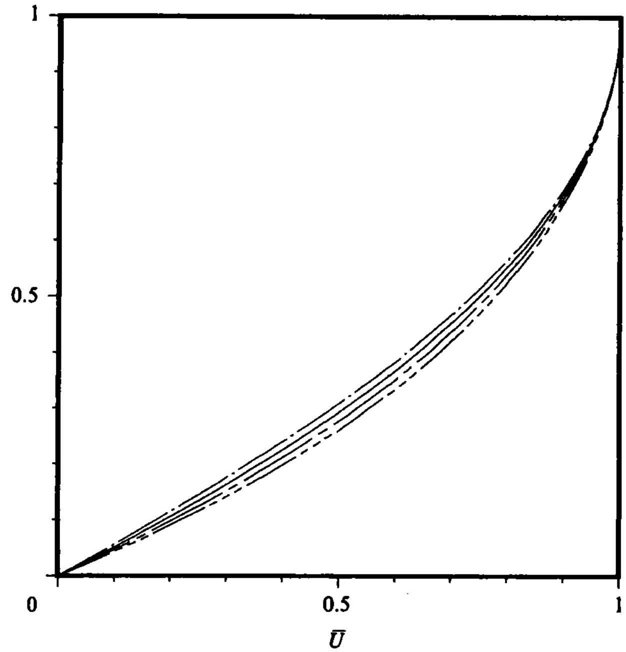


FIGURE 1. Dimensionless mean-velocity profiles. Form factors from left to right are FF = 2.57, 2.50, 2.44 and 2.38.

flow can be numerically calculated, the present intent is to obtain a general idea of the effects of form factor on stability.

The Orr–Sommerfeld equation was solved numerically using a 100-step Thomas (1953) finite-difference technique on a Vax 11/780 using double-precision arithmetic. As a check on the calculations, near orthonormalization (Gersting & Jankowski 1972) and compound matrix (Ng & Reid 1980) methods were also employed. Kaufman's (1972) LZ method was used to obtain the initial eigenvalue estimates. Chin (1981) contains further information on the calculations. Only one surface wave was found, and only the lowest shear mode was found to become unstable. These two modes were investigated in detail, and results are discussed in the following section. The higher-shear modes were found to be as complex in nature as those in plane Poiseuille flow as discussed in Grosch & Salwen (1968).

Four profiles from (5) were chosen to test form-factor effects. The FF's of these profiles were 2.57, 2.5 (parabola), 2.44, and 2.38 corresponding to $a = 0.05$, 0.0, $-0.05$, and $-0.1$ respectively. The surface tension parameter $\theta$ assumed the values of 0, 2140, and 4280 (20 °C), while the angle of inclination was either 1°, 4°, or 90°. Plane Poiseuille flow was also investigated for comparison.

## 3. Numerical calculations

Figure 1 shows the mean-velocity profile variations. Fullness of the profile indicates lower form factor with FF = 2.5 referring to a parabolic shape.

Figure 2 shows the neutral curves of the surface mode for FF = 2.5, but different values of surface tension. Increasing surface tension is a stabilizing effect. Anshus (1972) calculated analytically a $\frac{1}{3}$ power law for the surface mode at high *Re*. As can be seen, the present calculations substantiate the law. Figure 3 presents constant-$c_{\mathrm{Ti}}$

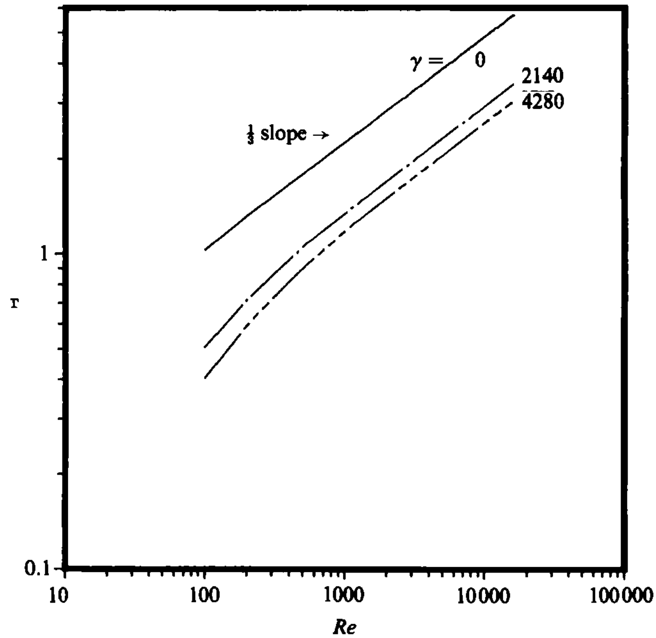


FIGURE 2. Gravity-mode neutral curves showing effects of varying angle and surface-tension parameter. FF = 2.50.

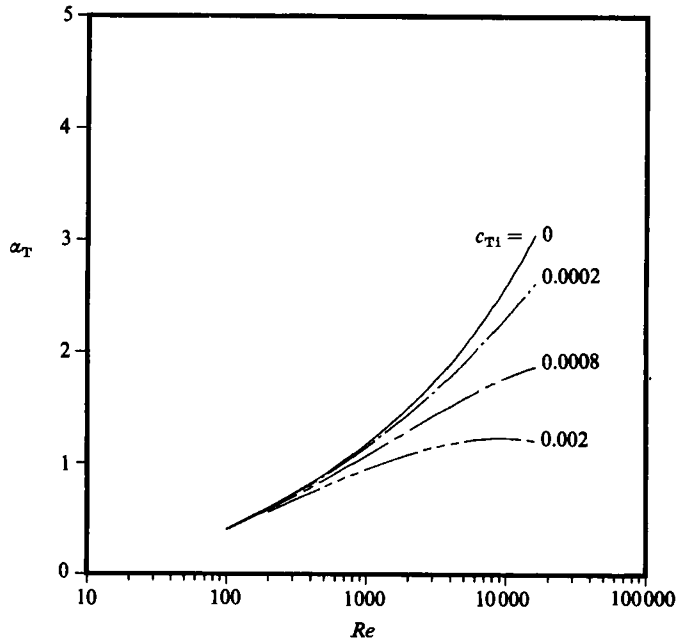


FIGURE 3. Constant $c_{Ti}$ curves of the gravity mode. FF = 2.50, $\theta = 4°$, $\gamma = 4280$.

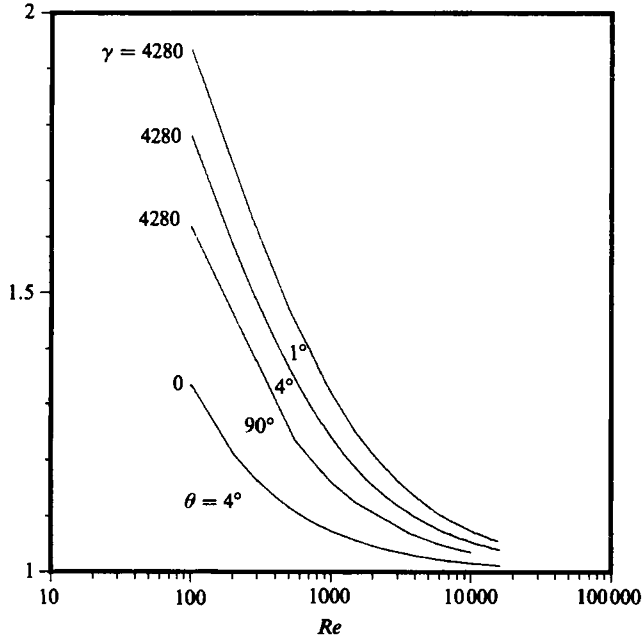


FIGURE 4. Gravity-mode wave velocities of neutral disturbances showing effects of angle and surface tension. FF = 2.50.

curves for a particular flow. Figure 4 shows the gravity-wave velocity of various neutral curves. The wave speed increases (decreases) as $\theta(\gamma)$ decreases. All curves initially start from $c_{Tr} = 2$ at the critical Reynolds number $Re_c$. At higher $Re$ they asymptotically approach the surface velocity. No critical layer, where wave speed equals fluid speed, exists. Form factor has little effect on the surface mode. Calculated neutral curves for $\gamma = 4280$, but different form factors, lie on top of the corresponding curve in figure 2. If the boundary conditions were allowed to take account of the streamwise curvature of the mean profile, little change would be expected because of the small effect on $Re_c$.

Figure 5 presents growth rates ($\alpha_T c_{Ti}$) versus $\alpha_T$ at a constant $Re$ of 2500. This is a typical value for the water-channel flows in Bertschy (1979). If one assumes that parabolic flow exists from the top of the table with $Q = 16.7\ \text{cm}^2/\text{s}$, $\nu = 1.0$ cs, then one obtains $H = 0.194$ cm and $U_s = 129$ cm/s. The maximum growth rate of figure 5 is approximately 4 $\text{s}^{-1}$. The wave velocity is about 130 cm/s so that the wave will travel the 180 cm length of the water channel in 1.5 s. In this time, the wave will have grown (barring nonlinear effects) to $e^6$ or 400 times its original size. However, no evidence of growing modes were seen on the water table owing to numerous calming devices.

Figure 6 presents neutral curves for the shear mode for three different angles. A plane Poiseuille flow neutral curve is also shown for comparison. Decreasing the angle from 90° to 1° results in a slight decrease in stability. At large $Re$, the branches for the plane-Poiseuille-flow curve follow $\frac{1}{7}$- and $\frac{1}{11}$-power laws (Lin 1954). From figures 6 and 7, it appears that similar asymptotes exist for the inclined-plane flow.

Figure 7 shows that increasing the surface-tension parameter from 0 to 4280 results in a slight destabilization. The merging of the curves at large $Re$ reflects the insensitivity of $\gamma$ in the normal-stress condition (3).

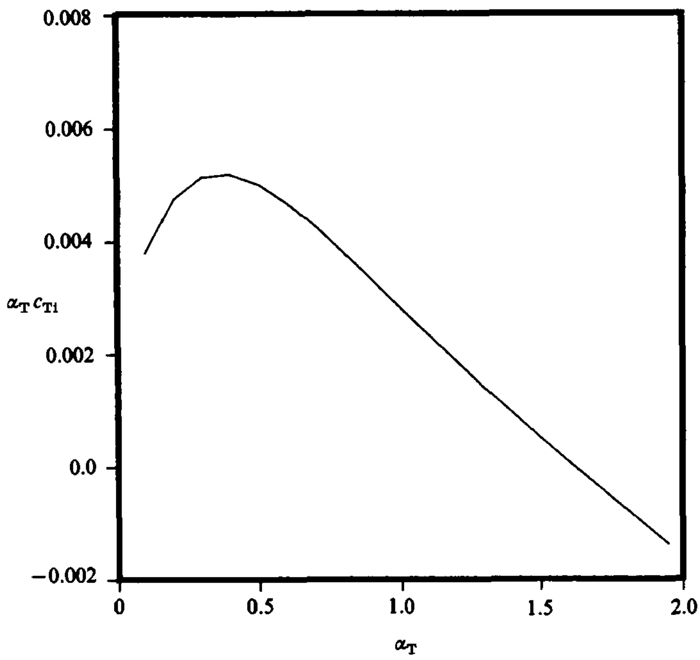


FIGURE 5. Gravity-mode amplification rates at $Re = 2500$, $\theta = 4°$, $\gamma = 4280$.

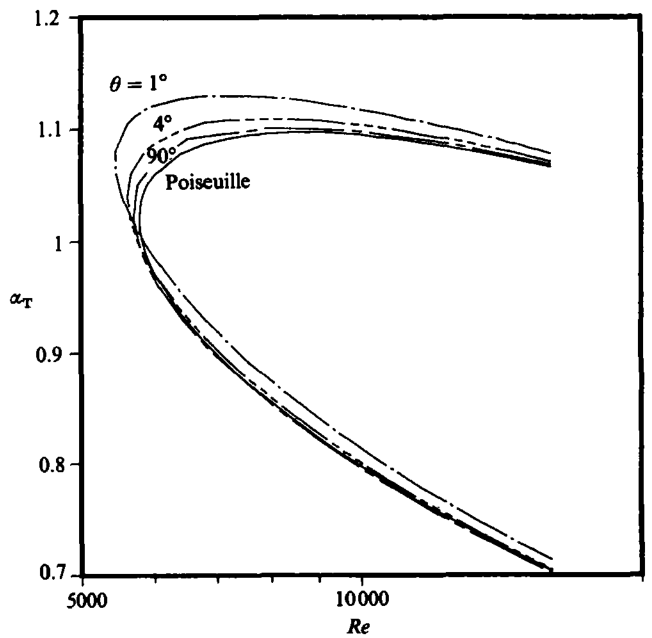


FIGURE 6. Shear-mode neutral curves showing effects of varying angle. Plane Poiseuille curve is also shown. FF = 2.50, $\gamma = 4280$.

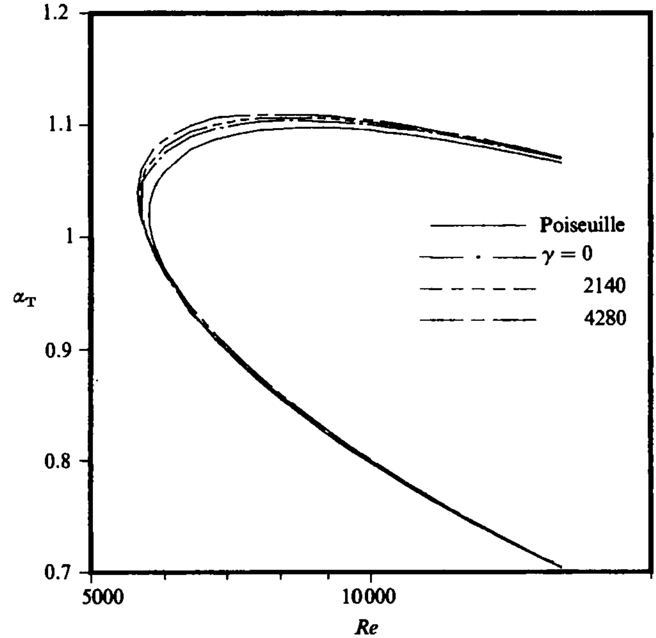


FIGURE 7. Shear-mode neutral curves showing effects of varying surface tension. FF = 2.50, $\theta = 4°$.

Figure 8 shows the shear wave velocities for two neutral curves. Change in angle and surface tension have little effect, as expected from previous figures. Note that the velocity is a double-valued function of *Re*, but always less than the surface velocity. A critical layer (where fluid velocity equals wave velocity) exists at which $c_{Tr} \approx 0.25$. Figure 9 presents the double-valued, constant-$c_{Ti}$ curves for the shear mode. The enormous effect of form factor is seen in figure 10. A 7 % decrease in FF increases $Re_c$ by an order of magnitude from 2200 to 20000. One can expect that the velocity profiles near the top of the water table have $Re_c$ in the hundreds of thousands. Hence, the water-table flow has decreasing stability as it proceeds down the glass.

In anticipation of experimental results, figure 11 presents temporal decay rates as a function of wavenumber for varying FF, but at a constant *Re*. As expected, there is increasing decay level with decreasing FF. There is a minimum in decay, while the wavenumber at this minimum remains approximately the same.

Mode shapes were also calculated. Figures 12 and 13 show the real and imaginary parts of $\phi(y)$ for both modes on their respective neutral curves for $\alpha_T = 1.04$. It is clear that the two mode shapes are different. The amplitude $u_A$ of the $u$-fluctuation is shown in figure 14 for the two modes on normalized coordinates. The phase of the $u$-fluctuation across the layer is shown in figure 15. There is no phase change in the flow down an inclined plane as there is in a Blasius boundary-layer flow.

Because of the interest in polymer solutions, some exploratory calculations were made with the Maxwell stress–strain relationship (Bird, Armstrong & Hassager 1977). It will suffice to state that increasing the fluid relaxation time results in destabilization for the shear mode. This was also found to be true for plane Poiseuille flow. More information on plane Poiseuille flow is contained in Porteous & Denn (1972), and Ho & Denn (1977). Destabilization is also predicted for the surface mode, following the calculations of Yih (1963).

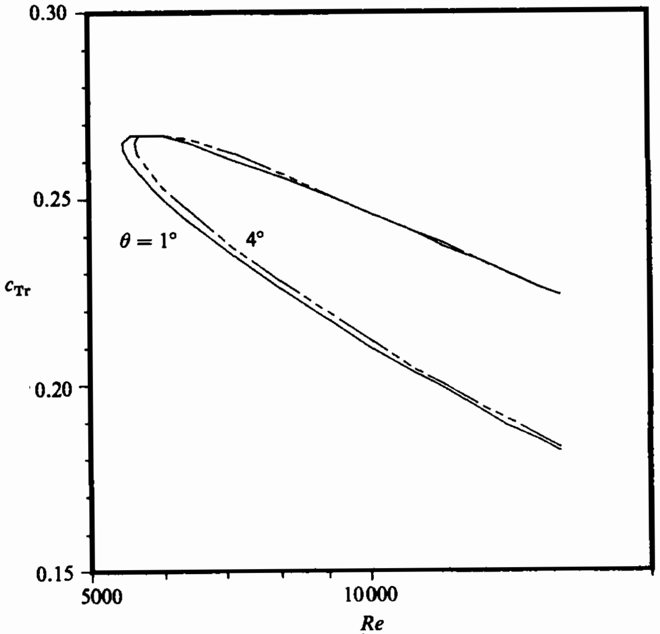


FIGURE 8. Shear-mode wave velocities of neutral disturbances. FF = 2.50, $\gamma = 4280$.

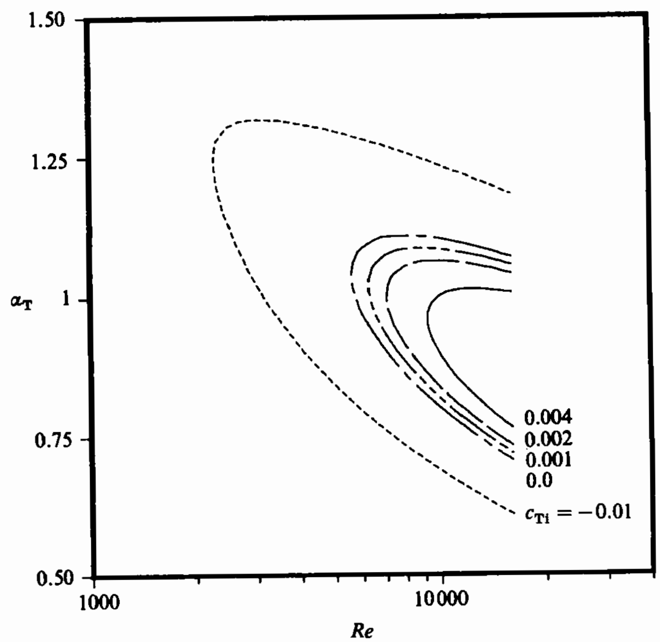


FIGURE 9. Constant-$c_{Ti}$ curves for the shear mode. FF = 2.50, $\theta = 4°$, $\gamma = 4280$.

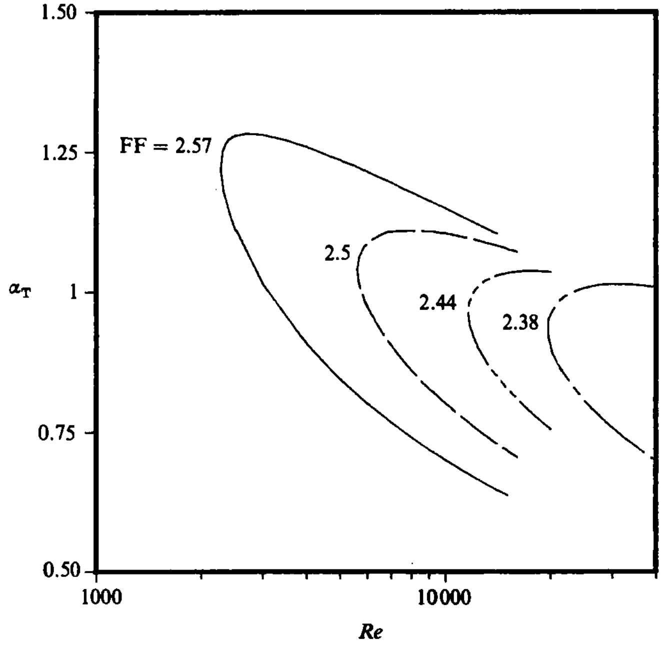


FIGURE 10. Shear-mode neutral curves showing effects of form factor. $\theta = 4°$, $\gamma = 4280$.

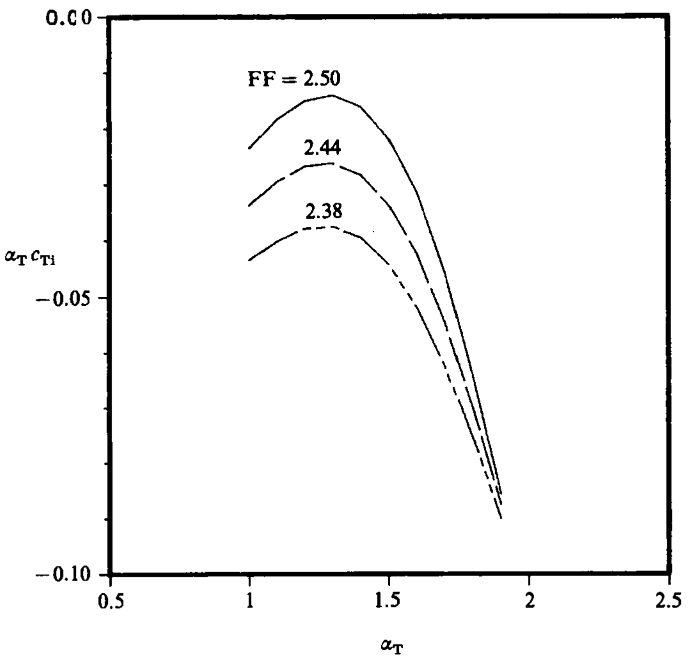


FIGURE 11. Decay rates of shear mode at $Re = 2000$, showing form-factor effects. $\theta = 4°$, $\gamma = 4280$.

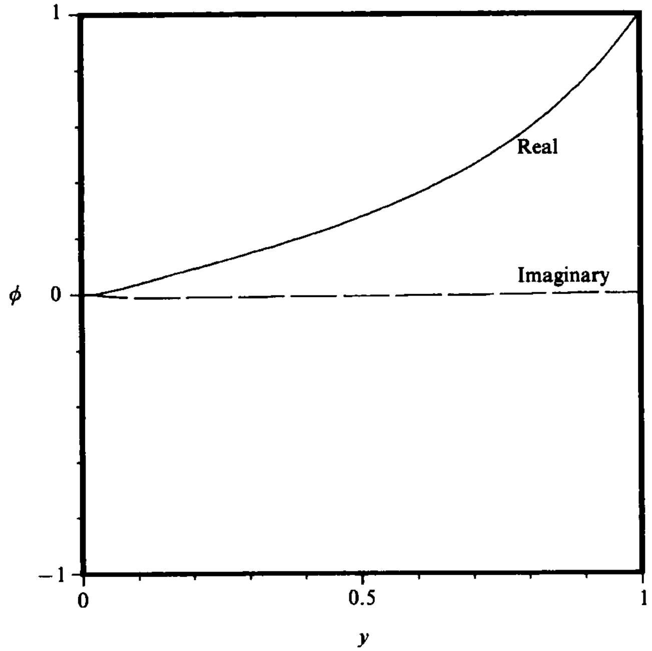


FIGURE 12. Gravity-mode eigenfunction profile. $c_T = 1.295 + 0.0i$, $\alpha_T = 1.04$, FF = 2.5, $\theta = 4°$, $\gamma = 4280$, $Re = 722$.

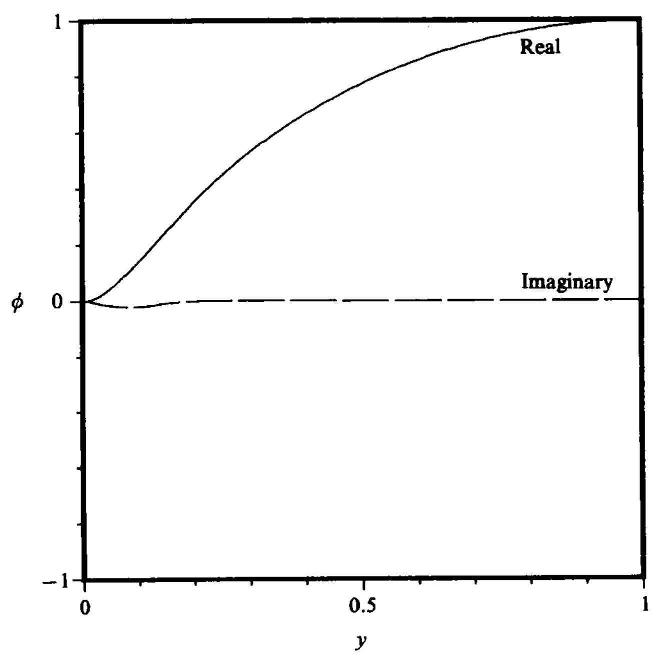


FIGURE 13. Shear-mode eigenfunction profile. $c_T = 0.2647 + 0.0i$, $\alpha_T = 1.04$, FF = 2.50, $\theta = 4°$, $\gamma = 4280$, $Re = 5606$.

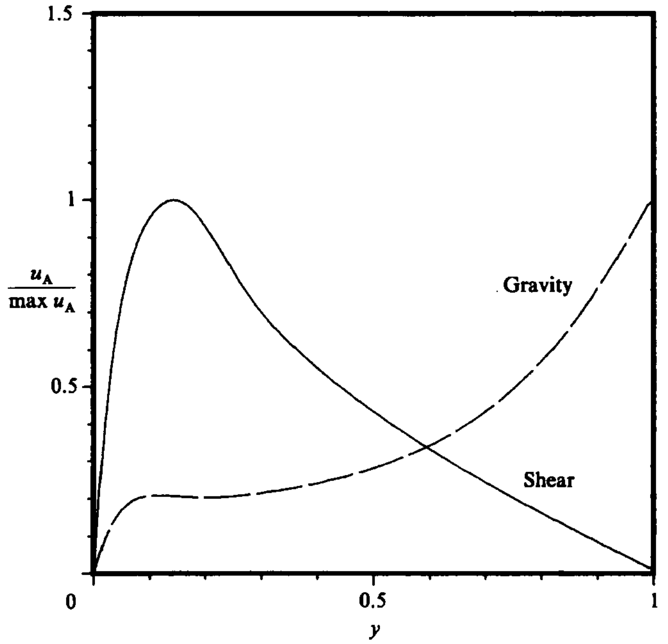


FIGURE 14. Amplitude of the $u$-fluctuation velocities determined from figures 12 and 13.

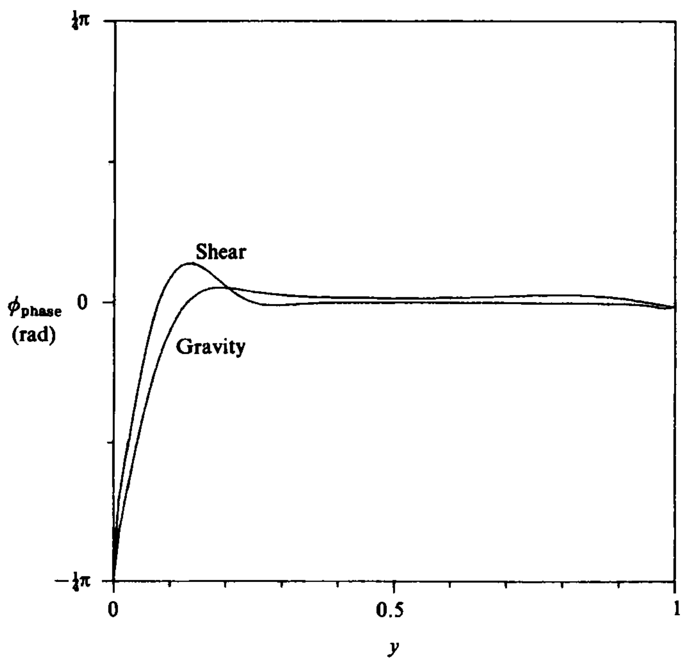


FIGURE 15. Phase profiles of the $u$-fluctuation velocities determined from figures 12 and 13.

It appears from previous calculations for both Newtonian and Maxwell fluids that, as far as the shear mode is concerned, one can consider the inclined plane flow to be plane Poiseuille. One can then surmise that Oldroyd, second-order as well as Maxwell, models will predict destabilization for the inclined plane flow (Kundu 1972; Porteous & Denn 1972).

On the other hand, one can draw a conclusion for the polymer solution flows from the form-factor effects. Drag-reducing polymer solutions, such as polyethylene oxide and polyacrylamide, are shear thinning fluids. The velocity provides of laminar polymer flows would then be fuller than those of corresponding water flows and hence have smaller form factors. These profiles would then be more stable. Of course, one must consider the extra terms in the Orr–Sommerfeld equation arising from the spatial dependence of viscosity. However, from the present calculations and those of Obremski, Morkovin & Landahl (1969), the form factor is the dominant parameter.

## 4. Summary

One can summarize the numerical calculations as follows. For the surface mode in the flow parameter range of interest ($\theta > 1°, 0 < \gamma < 5000$, $1000 < Re < 10{,}000$), we have that,

(i) the flow is moderately destabilized as $\theta$ increases,

(ii) the flow is stabilized by increasing surface tension, and

(iii) form factor has little effect.

For the shear mode, we conclude that,

(i) the flow is slightly stabilized as $\theta$ increases,

(ii) the flow is slightly destabilized as surface tension increases, and

(iii) form-factor effects are enormous.

REFERENCES

ABERNATHY, F. H., BERTSCHY, J. R., CHIN, R. W. & KEYES, D. E. 1980 Polymer induced fluctuations in high strain laminar flows. *J. Rheology* **24**, 647–665.

ANSHUS, B. E. 1972 On the asymptotic solution to the falling film problem. I and EC Fundamentals, **11**, 502–508.

BERTSCHY, J. R. 1979 Laminar and turbulent boundary layer flows of drag-reducing solutions. Ph.D. dissertation, Division of Applied Sciences, Harvard University.

BERTSCHY, J. R. & ABERNATHY, F. H. 1977 Modifications to laminar and turbulent boundary layer flows due to the addition of dilute polymer solutions. *2nd Intl Conf. on Drag Reduction, Paper* G1. BHRA Fluid Engineering.

BERTSCHY, J. R., CHIN, R. W. & ABERNATHY, F. H. 1983 High-strain rate free-surface boundary-layer flows. *J. Fluid Mech.* **126**, 443–461.

BIRD, R. B., ARMSTRONG, R. C. & HASSAGER, O. 1977 *Dynamics of Polymeric Liquids*. Wiley.

CHIN, R. W. 1981 Stability of flows down an inclined plane. Ph.D. dissertation, Division of Applied Sciences, Harvard University.

DE BRUIN, G. J. 1974 Stability of a layer of liquid flowing down an inclined plane. *J.Engng Maths* **8**, 259–270.

GASTER, M. 1962 A note on the relationship between temporally-increasing and spatially-increasing disturbances in hydrodynamic stability. *J. Fluid Mech.* **14**, 222–224.

GERSTING, J. M. & JANKOWSKI, D. F. 1972 Numerical methods for Orr–Sommerfeld Problems. *Intl J. Numer. Meth. Engng* **4**, 195–206.

GRAEF, M. 1966 Uber die Eigenschaften zwei- und dreidimensionaler Storungen in Rieselfilmen an geneigten Wanden. *Mitteilungen aus dem Max-Planck-Institut fur Stromungsforschung und der Aerodynamischen Versuchsanstalt*, vol. **36**, pp. 1–109.

GROSCH, C. E. & SALWEN, H. 1968 The stability of steady and time-dependent plane Poiseuille flow. *J. Fluid Mech.* **34**, 177–205.

HO, T. C. & DENN, M. M. 1977 Stability of plane Poiseuille flow of a highly elastic liquid. *J. Non-Newtonian Fluid Mech.* **3**, 179–195.

KAUFMAN, L. 1972 A generalization of the LR algorithm to solve $AX = \lambda BX$. *Stanford Computer Science Rep.* 276.

KUNDU, P. K. 1972 Small disturbance stability of plane Poiseuille flow of an Oldroyd fluid. *Phys. Fluids* **15**, 1207–1212.

LIN, C. C. 1954 *Theory of Hydrodynamic Stability.* Cambridge University Press.

LIN, S. P. 1967 Instability of a liquid film flowing down an inclined plane. *Phys. Fluids* **10**, 308–313.

NG, B. S. & REID, W. H. 1979 An initial value method for eigenvalue problems using compound matrices. *J. Comp. Phys.* **30**, 125–136.

OBREMSKI, H. J., MORKOVIN, M. V. & LANDAHL, M. 1969 A portfolio of stability characteristics of incompressible boundary layers. *AGARDograph* 134.

PIERSON, R. W. & WHITAKER, S. 1974 Some theoretical and experimental observations of the wave structure of falling liquid films. *I & EC Fundamentals*, **16**, 401–408.

PORTEOUS, K. C. & DENN, M. M. 1972 Linear stability of plane Poiseuille flow of viscoelastic liquids. *Trans. Soc. Rheol.* **16**, 295–308.

SQUIRE, H. B. 1933 On the stability of three dimensional distribution of viscous fluid between parallel walls. *Proc. R. Soc. Lond.* A**142**, 621–628.

THOMAS, L. H. 1953 The stability of plane Poiseuille flow. *Phys. Rev.* **91**, 780–783.

YIH, C. S. 1963 Stability of liquid flow down an inclined plane. *Phys. Fluids* **6**, 321–334.